JOURNAL OF FUNCTIONAL ANALYSIS **24**, 397–425 (1977)

Topological Aspects of Algebras of Unbounded Operators

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Communicated by the Editors

Received December 16, 1975

The utilization of $\mathcal{DF}$-spaces of A. Grothendieck leads to natural topologies on $*$-algebras of unbounded operators. In this way, the origin of pathologies is clarified, and natural classes of $*$-algebras with good behavior are introduced. Examples and particular algebras of countable algebraic dimension are studied in the second part of the paper.

INTRODUCTION

Infinite-dimensional representations of a Lie group G give rise (by differentiation of unitary representations of G) to algebras of unbounded operators: more precisely, the enveloping algebra (viewed in the representation) consists of operators generally unbounded defined on the Gårding domain, dense invariant domain. Since, from the physical point of view, observables have to be chosen among essentially self-adjoint operators of this algebra [21], it seems reasonable, for a better analysis of the structure, to define on $*$-algebras, a topology, which must be a generalization of the classical situation of $\mathbb{C}^*$-algebras. Taking in account that Von Neumann algebras are $\mathbb{C}^*$-algebras dual of some Banach space, called the predual, we are lead for our study to introduce $\mathcal{DF}$-spaces (which are dual of Fréchet spaces). These aspects accentuate remarks of [13].

The general plan of this paper is to examine topologies associated with $*$-algebras of unbounded operators and then to study in details natural classes of $*$-algebras (as shown by examples).

Section 1 is devoted to topological considerations: the comparison of the λ-topology (topologically satisfying) with the ρ-topology (algebraically interesting) is of great importance.

Section 2 is devoted to certain algebras of countable algebraic dimension. Examples and counterexamples are then investigated, showing the interest of this topological approach.

We recall briefly the definitions and notations of [13]. Algebras we will consider in the following will be algebras of operators, acting in some Hilbert

ISSN 0022–1236

space. We call $*$-algebra, in a Hilbert space $\mathfrak{h}$ an involutive algebra $\mathcal{A}$ of operators, not necessarily bounded, all defined on a domain $\mathscr{D}$ dense in $\mathfrak{h}$, with the following properties:

1° For $A \in \mathcal{A}$, the adjoint operator A^* satisfies $\operatorname{Dom} A^* \supset \mathscr{D}$ and $A\mathscr{D} \subset \mathscr{D}$, $A^*\mathscr{D} \subset \mathscr{D}$;

2° For $A \in \mathcal{A}$, $B \in \mathcal{A}$, $f \in \mathscr{D}$, we have

$$(A + B)f = Af + Bf, \qquad (AB)(f) = A(Bf).$$

Moreover, $1 \in \mathcal{A}$.

Condition $A = A^*$ in the algebra $\mathcal{A}$ means that the operator is symmetric. An operator $A \in \mathcal{A}$ is called positive (written $A \geqslant 0$ or $A \in \mathcal{A}^+$) iff

$$(Ax, x) \geqslant 0 \qquad \text{for all} \quad x \in \mathscr{D}.$$

The relation $\leqslant$ is clearly an order relation on $\mathcal{A}$, and a linear map $\Phi: \mathcal{A} \to \mathscr{B}$ from a $*$-algebra $\mathcal{A}$ (acting in a Hilbert space $\mathfrak{h}$) into a $*$-algebra $\mathscr{B}$ (acting in a Hilbert space $\mathscr{H}$) is called positive iff $A \geqslant 0$ in $\mathcal{A}$ implies $\Phi(A) \geqslant 0$ in $\mathscr{B}$.

In this paper, for an operator $A \in \mathcal{A}$, $\bar{A}$ denotes the closure of A (the domain of A being $\mathscr{D}$).

1. Topological Concepts

The ρ-Topology

On each $*$-algebra $\mathcal{A}$ (acting in some Hilbert space $\mathfrak{h}$, with $\mathscr{D}$ dense domain invariant under operators of $\mathcal{A}$), we define the ρ-topology, as follows: Given $A \geqslant 0$, we introduce, for $T \in \mathcal{A}$, the quantity

$$\rho_A(T) = \sup_{x \in \mathscr{D}} \frac{|(Tx, x)|}{(Ax, x)} \qquad \left(\frac{\lambda}{0} = +\infty \text{ for } \lambda > 0\right).$$

This defines the normed linear space

$$\mathfrak{N}_A = \{T \in \mathcal{A};\ \rho_A(T) < +\infty\}$$

with norm $\|\ \|_A = \rho_A \mid \mathfrak{N}_A$, the canonical injection $i_A : \mathfrak{N}_A \to \mathcal{A}$ and, on $\mathcal{A}$, the final locally convex topology, relative to the applications i_A (for all $A \in \mathcal{A}^+$). We note that $\bigcup_{A \in \mathcal{A}^+} \mathfrak{N}_A = \sum_{A \in \mathcal{A}^+} \mathfrak{N}_A = \mathcal{A}$; moreover, the relation $0 \leqslant A \leqslant B$ implies that the injection $i_{A,B} : (\mathfrak{N}_A, \|\ \|_A) \to (\mathfrak{N}_B, \|\ \|_B)$ is continuous, with norm smaller than 1. Also, if $\mathfrak{N}_A$ is strictly included in $\mathfrak{N}_B$, $i_{A,B}(\mathfrak{N}_A)$ cannot be a dense subspace of $(\mathfrak{N}_B, \|\ \|_B)$, because the open ball $B + \omega$, with $\omega = \{T \in \mathfrak{N}_B;\ \rho_B(T) < \frac{1}{2}\}$ does not intersect $\mathfrak{N}_A$. The ρ-topology is well defined by choosing maps i_{A_j} ($j \in J$ directed set, $A_j \geqslant 0$) such that $\mathfrak{A} = \bigcup_{j\in J} \mathfrak{N}_{A_j}$ [3], and is separated since, for every $T \neq 0 \in \mathfrak{A}$, there exists $x \in \mathscr{D}$ such that $(Tx, x) \neq 0$ [10, p. 138].

For a continuous linear map $f : \mathfrak{A} \to \mathbb{C}$, we put, for $A \in \mathfrak{A}$

$$\| f \|_{A,\rho} = \sup\{f(T);\ T \in \mathfrak{A} \text{ with } \rho_A(T) \leqslant 1\}.$$

PROPOSITION 1.1. *Let $\mathfrak{A} = \bigcup_{i\in I} \mathfrak{N}_{A_i}$ (I directed, $A_i \geqslant 0$) be a $*$-algebra, and $f : \mathfrak{A} \to \mathbb{C}$ be a linear form.*

f is positive if and only if $\| f \|_{A_i,\rho} = f(A_i)$ for all $i \in I$.

Proof. We denote by $\mathfrak{A}_{\mathbb{R}}$ the real linear space of symmetric elements of $\mathfrak{A}$. Then, clearly, $\mathfrak{A} = \mathfrak{A}_{\mathbb{R}} \oplus i\, \mathfrak{A}_{\mathbb{R}}$, and from the continuity of $T \to T^*$, the above direct sum is a topological direct sum. Hence a linear form is continuous if and only if its restriction to $\mathfrak{A}_{\mathbb{R}}$ is continuous.

We remark that, for a hermitian linear form g on $\mathfrak{A}$ (this means that $g(T^*) = \overline{g(T)}$, or equivalently $g(T) \in \mathbb{R}$ for symmetric T) the following equality is true

$$\| g \|_{A,\rho} = \sup\{g(T);\ T \in \mathfrak{A}_{\mathbb{R}} \text{ with } \rho_A(T) \leqslant 1\}.$$

Because, for every $\epsilon > 0$, there exists $T \in \mathfrak{A}$, with $\rho_A(T) \leqslant 1$ such that

$$| g(T) | \geqslant \| g \|_{A,\rho} - \epsilon.$$

Multiplying T by a complex number of modulus 1, it can be assumed that $g(T) \geqslant 0$. Hence

$$| g(\tfrac{1}{2}(T^* + T)) | = \tfrac{1}{2} \,| g(T) + g(T^*) | = g(T) \geqslant \| g \|_{A,\rho} - \epsilon$$

and, from $\rho_A(\frac{1}{2}(T + T^*)) \leqslant 1$ follows our equality. The proposition is therefore a consequence of [12].

DEFINITION 1.1. A $*$-algebra $\mathfrak{A}$ is said to be countably dominated, iff there exists an increasing sequence of subspaces $\mathfrak{N}_{A_n}$ ($A_n \geqslant 0$. $n \in \mathbb{N}$) such that

$$\mathfrak{A} = \bigcup_{n\in N} \mathfrak{N}_{A_n}.$$

One can check that algebras of countable algebraic dimension are countably dominated.

DEFINITION 1.2. A $*$-algebra $\mathfrak{A}$ is said to be ρ-closed, iff there exists a decomposition $\mathfrak{A} = \bigcup_{i\in I} \mathfrak{N}_{A_i}$ (with $A_i \geqslant 0$, I directed) such that, for every $i \in I$, $\mathfrak{N}_{A_i}$ is a Banach space.

In this paper, we restrict ourselves essentially to countably dominated $*$-algebras. Endowed with the ρ-topology, they are bornological quasi-barreled

$\mathscr{D}\mathscr{F}$-spaces [10]. We refer to [13] for complements. For a ρ-closed $*$-algebra $\mathfrak{A}$, the bilinear map

$$(S, T) \in \mathfrak{A} \times \mathfrak{A} \to ST \in \mathfrak{A}$$

is continuous (since $\mathfrak{A}$ is a barreled $\mathscr{D}\mathscr{F}$-space; and see [10, p. 168]); moreover, for every $A \in \mathfrak{A}^+$, $\mathfrak{N}_A$ is a Banach space. Indeed, if $(T_p)_{p\in\mathbb{N}}$ is a Cauchy sequence in $(\mathfrak{N}_A, \| \|_A)$,

$$|((T_p - T_q)x, x)| \leqslant \epsilon(p, q)(Ax, x) \qquad \text{with} \quad \lim_{\substack{p\to\infty \\ q\to\infty}} \epsilon(p, q) = 0.$$

Hence, for a certain $i \in I$

$$|((T_p - T_q)x, x)| \leqslant M\epsilon(p, q)\,(A_i x, x) \qquad (\text{for some } M < +\infty).$$

By completeness of the Banach space $\mathfrak{N}_{A_i}$, there exists an operator $T\colon \mathscr{D} \to \mathscr{D}$ such that:

$$|((T_p - T)x, x)| \leqslant \epsilon(p)(A_i x, x) \qquad \text{with} \quad \lim_{p\to\infty} \epsilon(p) = 0.$$

Since, for every $x \in \mathscr{D}$, $(Tx, x) = \lim_{p\to\infty} (T_p x, x)$, by letting $p \to \infty$ in the first majoration, we deduce that $(\mathfrak{N}_A, \| \|_A)$ is complete. We recall that, for $x \in \mathscr{D}$, $y \in \mathfrak{h}$, the linear form on $\mathfrak{A}$

$$T \to (Tx, y)$$

is denoted $\omega_{x,y}$. If $y \in \mathscr{D}$, the linear form $\omega_{x,y}$ is continuous on $\mathfrak{A}$; this follows from polarization equality and from [13]. If $y \in \mathfrak{h}$, the linear form $\omega_{x,y}$ is a simple limit of continuous linear form ω_{x,y_n} $(y_n \in \mathscr{D})$, and is not necessarily continuous (except, for example, if $\mathfrak{A}$ is ρ-closed).

PROPOSITION 1.2. *Let $\mathfrak{A}$ be a countably dominated $*$-algebra, and $S \subset \mathfrak{A}$ a subset of $\mathfrak{A}$. The following assertions are equivalent*:

(i) *S is bounded for the ρ-topology.*

(ii) *There exists $A \geqslant 0$ such that*

$$|(Tx, x)| \leqslant (Ax, x) \qquad \textit{for all} \quad x \in \mathscr{D} \textit{ and all } T \in S.$$

Proof. The implication (ii) $\Rightarrow$ (i) is immediate. For the converse, we first note that, for $A_1 \geqslant 0$, $A_2 \geqslant 0$,

$$\mathfrak{N}_{A_1} + \mathfrak{N}_{A_2} \subset \mathfrak{N}_{A_1+A_2}.$$

By Proposition 5 of [10, p. 171], there exists an operator $A \geqslant 0$ such that

$$S \subset \bar{U},$$

where

$$U = \{T \in \mathcal{A};\ |(Tx, x)| \leqslant (Ax, x) \text{ for all } x \in \mathscr{D}\},$$

$\bar{U}$ standing for the closure of the set U, relative to the ρ-topology. Hence, for $x \in \mathscr{D}$,

$$U \subset \omega_{x,x}^{-1}(B(0, r)) \qquad (r = (Ax, x))$$

($B(0, r)$ being the closed ball of the complex plane, of radius r), and by [13],

$$\bar{U} \subset \omega_{x,x}^{-1}(B(0, r)),$$

which achieves the proof.

The λ-topology

Let $\mathcal{A}$ be a $*$-algebra. We introduce, for $A \in \mathcal{A}$, the normed space $\mathfrak{M}_A$

$$\mathfrak{M}_A = \left\{T \in \mathcal{A};\ \sup_{x \in \mathscr{D}} \frac{\| Tx \|}{\| Ax \|} < \infty\right\} \qquad (\lambda/0 = \infty \text{ for } \lambda > 0)$$

with norm

$$\lambda_A(T) = \sup_{x \in \mathscr{D}} \frac{\| Tx \|}{\| Ax \|} \qquad \text{for} \quad T \in \mathfrak{M}_A$$

and on $\mathcal{A}$ the final locally convex topology relative to the canonical injection j_A : $\mathfrak{M}_A \to \mathcal{A}$. We note that the spaces $\mathfrak{M}_A$ constitute a directed set, since

$$\mathfrak{M}_A + \mathfrak{M}_B \subset \mathfrak{M}_{A^*A+B^*B+A+B+1} \,.$$

As previously, this topology (called the λ-topology) is well defined by choosing operators $A_i \in \mathcal{A}$ such that $\mathcal{A} = \bigcup_{i \in I} \mathfrak{M}_{A_i}$. Let us note that the λ-topology is invariant under isomorphisms (i.e., $\lambda_A(T) = \lambda_{\Phi(A)}(\Phi(T))$). More precisely, if Φ: $\mathcal{A} \to \mathscr{B}$ is a homomorphism (positive multiplicative map), then

$$\| T_n x \| \leqslant \epsilon(n) \| Ax \| \qquad \text{for all} \quad x \in \mathscr{D}, \qquad \lim_{n \to \infty} \epsilon(n) = 0,$$

implies

$$(T_n{}^* T_n x, x) \leqslant \epsilon^2(n)\, (A^* A x, x)$$

and by [13]

$$(\Phi(T_n{}^* T_n) x, x) \leqslant \epsilon^2(n)\, (\Phi(A^* A) x, x),$$

that is

$$\| \Phi(T_n) x \| \leqslant \epsilon(n) \| \Phi(A) x \|.$$

Also, from the equivalence of assertions (a) and (b) (proof left to the reader)

(a) $\mathscr{A}$ can be written $\mathscr{A} = \bigcup_{i\in\mathbb{N}} \mathfrak{M}_{A_i}$ $(A_i \in \mathscr{A})$,

(b) $\mathscr{A}$ can be written $\mathscr{A} = \bigcup_{j\in\mathbb{N}} \mathfrak{N}_{B_j}$ $(B_j \geqslant 0,\ B_j \in \mathscr{A})$,

it is not ambiguous for a $*$-algebra $\mathscr{A}$ to be countably dominated.

DEFINITION 1.3. A $*$-algebra $\mathscr{A}$ is said to be λ-closed, iff there exists a decomposition $\mathscr{A} = \bigcup_{i\in I} \mathfrak{M}_{A_i}$ such that, for every $i \in I$, $\mathfrak{M}_{A_i}$ is a Banach space.

It follows that, for every $A \in \mathscr{A}$, $\mathfrak{M}_A$ is a Banach space.

THEOREM 1.1. *Let $\mathscr{A}$ be a ρ-closed $*$-algebra, countably dominated. Then, $\mathscr{A}$ is λ-closed, and the λ-topology agrees with the ρ-topology.*

Proof. Since $\mathscr{A}$ is countably dominated, there exist operators $A_i \geqslant 0$ $(i \in \mathbb{N})$ such that $\mathscr{A} = \bigcup_{i\in\mathbb{N}} \mathfrak{N}_{A_i}$ and we can assume that $A_i^2 = A_i^* A_i \geqslant 1$. Hence $\mathscr{A} = \bigcup_{i\in\mathbb{N}} \mathfrak{N}_{A_i^* A_i}$, which implies $\mathscr{A} = \bigcup_{i\in\mathbb{N}} \mathfrak{M}_{A_i}$.

It clearly suffices to show that the normed spaces $\mathfrak{M}_{A_i}$ are Banach spaces. Let $(T_p)_{p\in\mathbb{N}}$ be a Cauchy sequence in the space $(\mathfrak{M}_{A_i}, \lambda_{A_i})$: then

$$\|(T_p - T_q)x\| \leqslant \epsilon(p, q)\, \|A_i x\| \qquad (\lim_{\substack{p\to\infty \\ q\to\infty}} \epsilon(p, q) = 0).$$

This relation defines a linear operator $T: \mathscr{D} \to \mathfrak{h}$ such that

$$Tx = \lim_p T_p x, \qquad x \in \mathscr{D}.$$

But

$$|((T_p - T_q)x, x)| \leqslant \epsilon(p, q)(A_i^* A_i x, x)^{1/2} \leqslant \epsilon(p, q)(A_i^* A_i x, x)$$

shows that $(T_p)_{p\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathfrak{N}_{A_i^* A_i}$. Since strong convergence implies weak convergence, we get $T \in \mathfrak{N}_{A_i^* A_i}$. Therefore, $T\mathscr{D} \subset \mathscr{D}$, which shows that $\mathfrak{M}_{A_i}$ is complete. Now, the estimate

$$\|Tx\| \leqslant \|A_i x\|$$

implies

$$|(Tx, x)| \leqslant (A_i^* A_i x, x)^{1/2} \leqslant (A_i^* A_i x, x)$$

which means that injection $K_{A_i}: \mathfrak{M}_{A_i} \to \mathfrak{N}_{A_i^* A_i}$ is continuous. Corollary 1 [10, p. 147] etablishes the equality of the two topologies.

DEFINITION 1.4. Let $\mathscr{A}$ be a $*$-algebra, with $\mathscr{D}$ dense domain invariant under operators of $\mathscr{A}$. The domain $\mathscr{D}$ is said to be closed (or $\mathscr{A}$-closed) iff

$$\mathscr{D} = \bigcap_{A\in\mathscr{A}} \operatorname{Dom}(\bar{A}).$$

PROPOSITION 1.3. *Let $\mathcal{A} = \bigcup_{i\in I} \mathfrak{M}_{A_i}$ be a $*$-algebra, with domain $\mathscr{D}$. $\mathscr{D}$ is $\mathcal{A}$-closed if and only if $\mathscr{D} = \bigcap_{i\in I} \mathrm{Dom}(\bar{A}_i)$.*

Proof. By hypothesis, for $T \in \mathcal{A}$, there exists $M < +\infty$, $i \in I$ such that

$$\| Tx \| \leqslant M \| A_i x \| \qquad \text{for all} \quad x \in \mathscr{D}.$$

This implies $\mathrm{Dom}\ \bar{T} \supset \mathrm{Dom}\ \bar{A}_i$, which completes the proof.

We recall that each $*$-algebra $\mathcal{A}$ induces on its domain $\mathscr{D}$ a natural topology [19, p. 88] defined by seminorms

$$x \in \mathscr{D} \to \| Ax \| \qquad (A \in \mathcal{A}).$$

If $\mathcal{A}$ is countably dominated, then $\mathcal{A} = \bigcup_{i\in\mathbb{N}} \mathfrak{M}_{A_i}$, and the topology of $\mathscr{D}$ can therefore be defined by seminorms

$$x \in \mathscr{D} \to \| A_n x \|, \qquad n \in \mathbb{N},$$

thus, $\mathscr{D}$ is a Fréchet space if $\mathscr{D}$ is $\mathcal{A}$-closed [19, Lemma 2.6].

PROPOSITION 1.4. *Let $\mathscr{D}$ be the domain of a $*$-algebra $\mathcal{A} = \bigcup_{i\in I} \mathfrak{N}_{A_i}$, and $f: \mathscr{D} \to \mathbb{C}$ a continuous linear form on $\mathscr{D}$.*

Then, there exist $y \in \mathfrak{h}$, and $i \in I$ such that

$$f(x) = (A_i x, y)$$

for all $x \in \mathscr{D}$.

Proof. Since the topology of $\mathscr{D}$ is generated by seminorms $x \to \| A_i x \|$, the continuity of f means that there exists $C > 0$, $i \in I$ such that

$$| f(x) | \leqslant C \| A_i x \| \qquad \text{for all} \quad x \in \mathscr{D}.$$

Let $j \in I$ such that

$$A_i^* A_i \leqslant M A_j \qquad A_j \geqslant 1 \ (M < +\infty).$$

Then

$$| f(x) | \leqslant MC(A_j x, x)$$

expresses that f is a continuous linear form on the pre-Hilbert space $(\mathscr{D}, (\ |\ )_j)$ endowed with the scalar product

$$(x, y)_j = (A_j x, y), \qquad x, y \in \mathscr{D}.$$

Using the fact that the completion of $(\mathscr{D}, (\ |\ )_j)$ can be identified with a subspace of $\mathfrak{h}$, we get $f(x) = (A_j x, y)$ for some $y \in \mathfrak{h}$.

LEMMA 1.1. *Let $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathfrak{N}_{A_n}$ be a countably dominated $*$-algebra, with* *closed domain $\mathscr{D}$. For a closable operator S defined on $\mathscr{D}$, such that $S\mathscr{D} \subset \mathscr{D}$, we can find an integer $n \in \mathbb{N}$ (resp. $m \in \mathbb{N}$) such that*

$$\sup_{x \in \mathscr{D}} \frac{\| Sx \|}{\| A_n x \|} < +\infty \qquad \left(\text{resp. } \sup_{x \in \mathscr{D}} \frac{|(Sx, x)|}{(A_m x, x)} < +\infty\right).$$

Proof. It follows from hypothesis that $\mathscr{D}$ endowed with its natural topology is a Fréchet space. It is now easy to check that the graph of the map $S: \mathscr{D} \to \mathscr{D}$ is closed. The closed graph ensures that S is continuous, which implies, for some $n \in \mathbb{N}$,

$$\| Sx \| \leqslant M \| A_n x \| \qquad \text{for all} \quad x \in \mathscr{D}.$$

Then, by routine majorations, we deduce

$$|(Sx, x)| \leqslant M(A_m x, x) \qquad \text{for some} \quad m \in \mathbb{N}, \text{ and all } x \in \mathscr{D}.$$

PROPOSITION 1.5. *Let $\mathfrak{A}$ be a countably dominated $*$-algebra, with closed domain $\mathscr{D}$, and $\mathscr{B}$ a $*$-algebra with the same domain $\mathscr{D}$ such that*

$$\mathfrak{A} \subset \mathscr{B}$$

Then, every positive linear form f on $\mathfrak{A}$ can be extended to a linear form $\tilde{f}$ positive on $\mathscr{B}$.

Proof. By Lemma 1.1, $\mathfrak{A}$ is a cofinal subset of $\mathscr{B}$, and the proposition is a consequence of Theorem 1.6.4 of [12, p. 24]. In particular, if $\mathscr{L}(\mathscr{D})$ denotes the set of operators (generally unbounded) acting in the Hilbert space $\mathfrak{h}$, which satisfy

$$\begin{aligned} &\operatorname{Dom} A = \mathscr{D} \qquad && A\mathscr{D} \subset \mathscr{D}, \\ &\operatorname{Dom} A^* \supset \mathscr{D} && A^*\mathscr{D} \subset \mathscr{D}, \end{aligned}$$

we can choose in the last proposition $\mathscr{B} = \mathscr{L}(\mathscr{D})$, or $\mathscr{B} = \mathfrak{A}''$ ($\mathfrak{A}''$ being calculated in the algebra $\mathscr{L}(\mathscr{D})$).

PROPOSITION 1.6. *Let $\mathfrak{A}$ be a countably dominated $*$-algebra, with closed domain $\mathscr{D}$, and S be a subset of $\mathfrak{A}$. Then, the following statements are equivalent:*

(i) *S is bounded for the λ-topology.*

(ii) *There exists $A \in \mathfrak{A}$, such that, for every $T \in S$*

$$\| Tx \| \leqslant \| Ax \| \qquad \textit{for all} \quad x \in \mathscr{D}.$$

(iii) *S is bounded for $\mathfrak{A}$ endowed with topology defined by seminorms*

$$T \to \| Tx \| \qquad x \in \mathscr{D}.$$

(iv) *S is bounded for $\mathcal{A}$ endowed with topology defined by seminorms*

$$T \to |(Tx, y)| \qquad x \in \mathscr{D}, y \in \mathfrak{h}.$$

Proof. It suffices to establish the implication (iv) $\Rightarrow$ (ii) By the Banach–Steinhaus theorem, it follows from (iv) that, for every $x \in \mathscr{D}$

$$\sup_{T \in S} \| Tx \| < +\infty.$$

As in Lemma 1.1, we can apply the closed graph theorem to operators T of S, viewed as a linear map from $\mathscr{D}$ endowed with its natural topology into the Hilbert space $\mathfrak{h}$. The preceding estimate shows that S is a simply bounded subset of $\mathscr{L}_s(\mathscr{D}, \mathfrak{h})$. Thus, the equicontinuity principle implies that the set $S \subset \mathscr{L}_s(\mathscr{D}, \mathfrak{h})$ is equicontinuous, which gives (ii).

COROLLARY 1.1. *Let $\mathcal{A}$ be a countably dominated $*$-algebra, with closed domain $\mathscr{D}$, endowed with the λ-topology. Then, assertions* (i) *and* (ii) *are equivalent*:

(i) *For every $S \in \mathcal{A}$, the map $T \to ST$ is continuous.*

(ii) *The bilinear map $(S, T) \to ST$ is continuous.*

Let us mention that the map $T \to TS$ is always λ-continuous, because the relation $\| Tx \| \leqslant \| Ax \|$ (for all $x \in \mathscr{D}$) implies $\| TSx \| \leqslant \| ASx \|$.

Proof. By virtue of [10, p. 162], we establish (ii) $\Rightarrow$ (i) by showing that $(S, T) \to ST$ is hypocontinuous. We put $\pi(S)T = ST$. Let S be a bounded subset of $\mathcal{A}$; by the preceding result, one can find $A \in \mathcal{A}$ such that for every $T \in S$,

$$\| Tx \| \leqslant \| Ax \| \qquad \text{for all} \quad x \in \mathscr{D}.$$

The equicontinuity of maps $\pi(T)$ $(T \in S)$ amounts to show that maps $\pi(T)$: $\mathfrak{M}_B \to \mathcal{A}$ are equicontinuous, for every $B \in \mathcal{A}$. Since $\pi(A)$ is continuous, the unit ball of the normed space $(\mathfrak{M}_B, \lambda_B)$ is sent into a unit ball of some $\mathfrak{M}_C$ $(C \in \mathcal{A})$, which means that

$$\| Rx \| \leqslant \| Bx \| \text{ for all } x \in \mathscr{D} \text{ implies } \| ARx \| \leqslant \| Cx \|.$$

But $\| TRx \| \leqslant \| ARx \|$ for all $x \in \mathscr{D}$. Thus, the unit ball of $\mathfrak{M}_B$ is sent by operators $\pi(T)$, $T \in S$, into a unit ball of some $\mathfrak{M}_C$. This achieves our proof.

One can show that under the hypotheses of Corollary 1.1, the completion of the normed spaces $\mathfrak{M}_B$ $(B \in \mathcal{A})$ can be identified with operators S such that $S\mathscr{D} \subset \mathscr{D}$. We thus obtain a bigger space $\tilde{\mathcal{A}}$, which is an algebra (with natural multiplication).

THEOREM 1.2. *Let $\mathcal{A}$ be a countably dominated $*$-algebra, with closed domain $\mathscr{D}$. The following statements are equivalent*:

(i) *The λ-topology is identical to the ρ-topology.*

(ii) *The bounded subsets of the λ-topology coincide with the bounded subsets of the ρ-topology.*

(iii) *The linear forms $\omega_{x,y}$ ($x \in \mathscr{D}$, $y \in \mathfrak{h}$) are ρ-continuous.*

(iv) *For every $A \in \mathcal{Q}$, there exists $B \in \mathcal{Q}$ such that $\mathfrak{N}_B \subset \mathfrak{M}_B$ and the injection i: $\mathfrak{N}_A \to \mathfrak{M}_B$ is continuous.*

(v) *The bilinear map $(S, T) \to ST$ is ρ-continuous.*

Proof. For simplicity of notations, we denote by $(\mathcal{Q}, \lambda)$ resp. $((\mathcal{Q}, \rho))$ the space $\mathcal{Q}$ endowed with the λ-topology (resp. the ρ-topology). The equivalence of (i) and (ii) follows from the fact that $(\mathcal{Q}, \lambda)$ and $(\mathcal{Q}, \rho)$ are bornological spaces.

Clearly, (i) implies (iii). Now, if S is a bounded subset of $(\mathcal{Q}, \rho)$, and if (iii) holds, then the injection i: $(\mathcal{Q}, \rho) \to (\mathcal{Q}, \mathscr{T})$ ($\mathscr{T}$ topology on $\mathcal{Q}$ defined by seminorms $T \to (Tx, y)$ $x \in \mathscr{D}$, $y \in \mathfrak{h}$) is continuous; hence, S is bounded for $\mathscr{T}$, and by Proposition 1.2, S is bounded for $(\mathcal{Q}, \lambda)$. This gives (iii) $\Rightarrow$ (ii).

The equivalence of (iv) and (ii) is obvious, and the equivalence of (i) and (v) follows from Corollary 1.1 and from ρ-continuity of $T \to T^*$.

Of course, if $\mathcal{Q}$ consists of bounded operators, $(\mathcal{Q}, \lambda)$ is equal to $(\mathcal{Q}, \rho)$, and these topologies agree with the norm topology.

DEFINITION 1.5. Let $\mathcal{Q}$ be a $*$-algebra, with dense domain $\mathscr{D}$. A sequence $(x_i)_{i \in \mathbb{N}}$ of elements of $\mathscr{D}$ is called σ-convergent, iff, for every $A \in \mathcal{Q}$,

$$\sum_{i=1}^{\infty} \|Ax_i\|^2 < +\infty. \tag{1}$$

If $\mathcal{Q} = \bigcup_{j \in J} \mathfrak{M}_{A_j}$, then a sequence (x_i) of elements of $\mathscr{D}$ is σ-convergent if and only if (1) holds for operators A_j ($j \in J$). Moreover, if $\mathscr{D}$ is $\mathcal{Q}$-closed, then for every closable operator S such that $S\mathscr{D} \subset \mathscr{D}$, the relation (1) holds, with $A = S$: this is a consequence of Lemma 1.1, and this shows that this definition is not related to the algebraic structure of $\mathcal{Q}$.

But this concept leads to a natural definition of the ampliation $\mathcal{Q} \otimes \mathbb{1}_R$ of a $*$-algebra $\mathcal{Q}$. In fact, let R be a (separable) Hilbert space. We identify the Hilbert space $\mathfrak{h} \otimes R$ to $\bigoplus_{n \geq 0} \mathfrak{h}_n$ (with $\mathfrak{h}_n = \mathfrak{h}$ for every $n \in \mathbb{N}$) and we define, for $T \in \mathcal{Q}$, the operator $T \otimes 1_R$ to be

$$T \otimes 1_R = \bigoplus_{n \geq 0} T_n \qquad \text{with} \quad T_n = T \qquad \text{for every} \quad n$$

and

$$(T \otimes 1_R)(\mathscr{D}_n) \subset \mathfrak{h}_n\,.$$

Thus, every operator of the form $T \otimes 1$ ($T \in \mathcal{Q}$) is defined on a subspace containing the algebraic direct sum $\bigoplus_{n \geq 0} \mathscr{D}_n$ (with $\mathscr{D}_n = \mathscr{D} \subset \mathfrak{h}_n$). Clearly, every σ-convergent sequence can be viewed as an element of $\mathfrak{h} \otimes R$ (take $A = \mathrm{Id}$ in relation (1)) and under this identification, the set $\tilde{D}$ of σ-convergent sequences appears as the intersection of the domains of the closure of operators $T \otimes 1$ defined on the algebraic sum $\oplus_{n>0} \mathscr{D}_n$ ($\mathscr{D}$ being $\mathcal{A}$-closed). Therefore, $\tilde{D}$ is invariant under operators $T \otimes 1$, $\tilde{D}$ is $(\mathcal{A} \otimes \mathbb{1})$-closed, and the map $T \to T \otimes 1$ is an isomorphism of $\mathcal{A}$ onto $\mathcal{A} \otimes \mathbb{1}$.

This leads to

DEFINITION 1.6. Let $\mathcal{A}$ be a $*$-algebra, with closed domain $\mathscr{D}$. We introduce, on $\mathcal{A}$:

(i) the strong topology, generated by seminorms

$$T \in \mathcal{A} \to \| Tx \| \qquad x \in \mathscr{D};$$

(ii) the weak topology, generated by seminorms

$$T \in \mathcal{A} \to (Tx, y) \qquad x \in \mathscr{D},\ y \in \mathfrak{h},$$

(iii) the σ-strong topology, generated by seminorms

$$T \in \mathcal{A} \to \left(\sum_{i=1}^{\infty} \| Tx_i \|^2 \right)^{1/2}$$

$(x_i)_{i \in \mathbb{N}}$ belonging to the set of σ-convergent sequences,

(iv) the σ-weak topology, generated by seminorms

$$T \in \mathcal{A} \to \sum_{i=1}^{\infty} (Tx_i, y_i),$$

$(x_i)_{i \in \mathbb{N}}$ belonging to the set of σ-convergent sequences, and $y_i \in \mathfrak{h}$ satisfying $\sum_{i=1}^{\infty} \| y_i \|^2 < +\infty$.

THEOREM 1.3. *Let $\mathcal{A}$ be a $*$-algebra, with closed domain $\mathscr{D}$, and let φ be a linear form on $\mathcal{A}$.*

(i) *The following statements are equivalent.*

(i.1) *φ is weakly continuous,*

(i.2) *φ is strongly continuous,*

(i.3) *$\varphi = \sum_{i \in I} \omega_{x_i, y_i}$, with $x_i \in \mathscr{D}$, $y_i \in \mathfrak{h}$, I finite.*

(ii) *The following statements are equivalent.*

(ii.1) *φ is σ-weakly continuous,*

(ii.2) *φ is σ-strongly continuous,*

(ii.3) *$\varphi = \sum_{i \in \mathbb{N}} \omega_{x_i, y_i}$, with $x_i \in \mathscr{D}$, $y_i \in \mathfrak{h}$, $\sum_{i \in \mathbb{N}} \| y_i \|^2 < +\infty$, and $(x_i)_{i \in \mathbb{N}}$ being a σ-convergent sequence.*

Proof. We first prove part (i). Since the set of linear forms of the type $\varphi = \sum_{i\in I} \omega_{x_i, y_i}$ (I finite, $x_i \in \mathscr{D}$, $y_i \in \mathfrak{h}$) is a vector subspace F of $\mathcal{O}\!\!\!\mathcal{l}^*$, algebraic dual of $\mathcal{O}\!\!\!\mathcal{l}$, the weak topology of $\mathcal{O}\!\!\!\mathcal{l}$ is the topology $\sigma(\mathcal{O}\!\!\!\mathcal{l}, F)$, and therefore (i.3) is equivalent to (i.1). Of course, (i.3) implies (i.2). Conversely, let φ be a strongly continuous linear form. Then, for some finite family $(x_i)_{1\leqslant i\leqslant n}$ of elements of $\mathscr{D}$,

$$|\varphi(T)| \leqslant \left(\sum_{i=1}^{n} \| Tx_i \|^2\right)^{1/2} \qquad \text{for all} \quad T \in \mathcal{O}\!\!\!\mathcal{l}.$$

Assume first $n = 1$, that is $|\varphi(T)| \leqslant \| Tx \|$ $(x \in \mathscr{D})$. Then the linear map $\tilde{\varphi}: \mathcal{O}\!\!\!\mathcal{l}x \to \mathbb{C}$ defined by

$$\tilde{\varphi}(Tx) = \varphi(T)$$

is well defined (since $Tx = Sx$ implies $\varphi(T - S) = 0$), and satisfies

$$|\tilde{\varphi}(y)| \leqslant \| y \| \qquad \text{for all} \quad y \in \mathcal{O}\!\!\!\mathcal{l}x.$$

By the Hahn–Banach theorem, $\tilde{\varphi}$ has a continuous extension (denoted again $\tilde{\varphi}$) such that $|\tilde{\varphi}(y)| \leqslant \| y \|$ for all $y \in \mathfrak{h}$.

Thus $\tilde{\varphi}(y) = (y, z)$ for some $z \in \mathfrak{h}$, and therefore $\varphi(T) = (Tx, z)$ for all $T \in \mathcal{O}\!\!\!\mathcal{l}$.

In the case $n \geqslant 1$, we consider the ampliation map

$$\Phi: \mathcal{O}\!\!\!\mathcal{l} \to \mathcal{O}\!\!\!\mathcal{l} \otimes \mathbb{1}_{\mathbb{C}^n}.$$

Then, putting $\theta = \varphi \circ \Phi^{-1}$, $\tilde{x} = (x_i)_{1\leqslant i\leqslant n}$, we get

$$|\theta(T \otimes 1)| \leqslant \|(T \otimes 1)\tilde{x}\|$$

which gives our result.

Part (ii) follows from part (i), because, if $(x_i)_{i\in\mathbb{N}}$ is a σ-convergent sequence, then, for every $T \in \mathcal{O}\!\!\!\mathcal{l}$

$$\left(\sum_{i=1}^{\infty} \| Tx_i \|^2\right)^{1/2} = \|(T \otimes 1_R)\tilde{x}\|,$$

$$\sum_{i=1}^{\infty} (Tx_i, y_i) = (T \otimes 1_R\tilde{x}, \tilde{y}),$$

with $R = L_{\mathbb{C}}^2(\mathbb{N})$, $\tilde{x} = (x_i)_{i\in\mathbb{N}}$, $\tilde{y} = (y_i)_{i\in\mathbb{N}}$.

COROLLARY 1.2. *Every weakly (resp. σ-weakly) continuous linear form on $\mathcal{O}\!\!\!\mathcal{l}$ has a weakly (resp. σ-weakly) continuous extension to $\mathscr{L}(\mathscr{D})$.*

This is a consequence of a remark following Definition 1.5.

Let $\mathfrak{A} = \bigcup_{i\in\mathbb{N}} \mathfrak{M}_{A_i}$ be a $*$-algebra, with closed domain $\mathscr{D}$. We endow $\mathfrak{A}$ with its λ-topology, and denote $\mathfrak{A}'$ the strong dual of $(\mathfrak{A}, \lambda)$. For $f \in \mathfrak{A}'$, $n \in \mathbb{N}$, we put

$$\|f\|_{\lambda, A_n} = \sup\{|f(T)|;\ T \in \mathfrak{M}_{A_n} \text{ with } \lambda_{A_n}(T) \leqslant 1\}.$$

By [10], the strong topology of $\mathfrak{A}'$ is the coarsest for which the transposed mappings ${}^t i_{A_i}$ of $\mathfrak{A}'$ into the strong duals $(\mathfrak{M}'_{A_i}, \|\ \|_{\lambda, A_i})$ are continuous.

We denote by

(a) $\mathscr{L}_\sim$ the space of weakly continuous (equivalently strongly continuous) linear forms on $\mathfrak{A}$. Of course, $\mathscr{L}_\sim \subset \mathfrak{A}'$,

(b) $\mathscr{L}_*$ the adherence of $\mathscr{L}_\sim$ in $\mathfrak{A}'$ (endowed with its strong topology).

THEOREM 1.4. 1° *Every σ-weakly continuous linear form on $\mathfrak{A}$ belongs to $\mathscr{L}_*$.*

2° *Conversely, if the algebra $\mathfrak{A}$ satisfies condition* (C), *then every element of $\mathscr{L}_*$ is σ-weakly continuous.*

Condition (C) *is*:

There exists an operator $\Delta \in \mathfrak{A}$, which is the restriction to $\mathscr{D}$ of the inverse of a compact operator in $\mathfrak{h}$, such that

$$\mathfrak{A} = \bigcup_{n\in\mathbb{N}} \mathfrak{M}_{\Delta^n}.$$

Before plunging into the proof of this theorem, let us note that if we differentiate a strongly continuous unitary irreducible representation π of a Lie group G (with Lie algebra $\mathfrak{g}$), the enveloping algebra $\mathfrak{A}$ generated by $d\pi(\mathfrak{g})$ admits the domain $\mathscr{D}$ of differentiable vectors as closed domain and

$$\mathfrak{A} = \bigcup_{n\in\mathbb{N}} \mathfrak{M}_{(1-\Delta)^n} \qquad (\Delta \text{ Laplacien: [16]}).$$

Moreover, (C) is fulfilled if G is C.C.R. [17].

The proof rests on

LEMMA 1.2. (adaptation of Lemma 2 of [23]). *The vector space of operators of $\mathscr{L}(\mathscr{D})$ with finite dimensional range, is dense in $\mathscr{L}(\mathscr{D})$ endowed with the λ-topology.*

A similar proof of Lemma 3 of [4, p. 36], and Lemma 1.2 establish

LEMMA 1.3. *Let $(x_j)_{1\leqslant j\leqslant p}$ (resp. $(x_k')_{1\leqslant k\leqslant q}$) be vectors of $\mathscr{D}$, and $(y_j)_{1\leqslant j\leqslant p}$ (resp. $(y_k')_{1\leqslant k\leqslant q}$) be vectors of $\mathfrak{h}$ such that the linear maps from $\mathfrak{h}$ into $\mathscr{D}$*

$$y \to \sum_{j=1}^{p} (y|y_j)\, x_j$$

$$y \to \sum_{k=1}^{q} (y|y_k')\, x_k'$$

coincide. Then, for every $T \in \mathscr{L}(\mathscr{D})$

$$\sum_{j=1}^{p} (Tx_j, y_j) = \sum_{k=1}^{q} (Tx_k', y_k').$$

Proof of the theorem. We first treat part 1°. Let $\varphi = \sum_{i\in\mathbb{N}} \omega_{x_i,y_i}$ be a σ-weakly continuous linear form on $\mathfrak{A}$ ($(x_i)_{i\in\mathbb{N}}$ is a σ-convergent sequence, and $\sum_{i=1}^{\infty} \| y_i \|^2 < +\infty$). Let $\varphi_n = \sum_{i=1}^{n} \omega_{x_i,y_i}$. Then $\varphi_n \in \mathscr{L}_\sim$, and φ_n tends to φ in $\mathfrak{A}'$ because, for every $A \in \mathfrak{A}$,

$$\| \varphi_n - \varphi \|_{A,\lambda} \to 0$$

since the relation

$$\| Tx \| \leqslant \| Ax \|$$

implies

$$\left| \sum_{i=n+1}^{\infty} \omega_{x_i,y_i}(T) \right| \leqslant \sum_{i=n+1}^{\infty} \| Ax_i \| \| y_i \| \leqslant \tfrac{1}{2} \sum_{i=n+1}^{\infty} (\| Ax_i \|^2 + \| y_i \|^2).$$

For part 2°, it suffices to consider the case $\mathfrak{A} = \mathscr{L}(\mathscr{D})$ because of Corollary 1.2 and the relation

$$\sup\{| f(T) |;\ T \in \mathfrak{A} \text{ such that } \| Tx \| \leqslant \| Ax \|\}$$
$$\leqslant \sup\{| f(T) |;\ T \in \mathscr{L}(\mathscr{D}) \text{ such that } \| Tx \| \leqslant \| Ax \|\},$$

for $f \in \mathfrak{A}'$.

We can assume with no loss of generality that $\| \Delta x \| \geqslant \| x \|$ (for all $x \in \mathscr{D}$). Thus

$$0 \leqslant p \leqslant n \Rightarrow \| \Delta^p x \| \leqslant \| \Delta^n x \|$$

for all $x \in \mathscr{D}$.

Let $\varphi \in \mathscr{L}_*$. One can find elements $\varphi_k \in \mathscr{L}_\sim$ such that

$$\| \varphi - \varphi_1 \|_{\Delta^2,\lambda} \leqslant 2^{-1}$$

$$\| \varphi - \varphi_1 - \varphi_2 \|_{\Delta^3,\lambda} \leqslant 2^{-2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots \leqslant \left\| \varphi - \sum_{i=1}^{n} \varphi_i \right\|_{\Delta^n,\lambda} \leqslant \left\| \varphi - \sum_{i=1}^{n} \varphi_i \right\|_{\Delta^{n+1},\lambda} \leqslant 2^{-n}.$$

Thus, $\varphi = \sum_{k=1}^{\infty} \varphi_k$, with for every $n \in \mathbb{N}$, $\sum_{k=1}^{\infty} \| \varphi_k \|_{\Delta^n,\lambda} < +\infty$ since $\| \varphi_k \|_{\Delta^k,\lambda} \leqslant 2^{-k}$. By hypothesis

$$\varphi_k = \sum_{j=1}^{p_k} \omega_{x_j^k, y_j^k}$$

with $x_j^k \in \mathscr{D}$, $y_j^k \in \mathfrak{h}$. For simplicity of notation, we omit the index k.

The polar decomposition of the map

$$y \to \sum_{j=1}^{pk} (y, y_j^k)\, \Delta^k x_j^k = \sum_{j=1}^{p} (y, y_j)\, \Delta^k x_j$$

shows that we can write

$$\sum_{j=1}^{p} (y, y_j)\, \Delta^k x_j = \sum_{i=1}^{q=q_k} \lambda_i (y, e_i)\, \Delta^k e_i'$$

with $e_i' \in \mathscr{D}$, $e_i \in \mathfrak{h}$, $\lambda_i \geqslant 0$, the system of $(e_i)_{1\leqslant i\leqslant q}$ (resp. $(\Delta^k e_i')_{1\leqslant i\leqslant q}$) being orthonormal.

By injectivity of Δ^k, and Lemma 1.3, we deduce that

$$\sum_{j=1}^{p} \omega_{x_j, y_j} = \sum_{i=1}^{q} \lambda_i \omega_{e_i', e_i}$$

Now,

$$\Big\| \sum_{i=1}^{q} \lambda_i \omega_{e_i', e_i} \Big\|_{\Delta^k, \lambda} = \sum_{i=1}^{q} \lambda_i \,. \tag{2}$$

Indeed, the relation $\| Tx \| \leqslant \| \Delta^k x \|$, $(T \in \mathscr{L}(\mathscr{D}))$ for all $x \in \mathscr{D}$ implies

$$\Big| \sum_i \lambda_i \omega_{e_i', e_i}(T) \Big| \leqslant \sum_i \lambda_i \| \Delta^k e_i' \| \| e_i \| = \sum_i \lambda_i \,.$$

If we define the operator H_k by

$$H_k e_j' = e_j \qquad 1 \leqslant j \leqslant q$$

then

$$\| H_k x \|^2 = \sum_{i,j} x_i \bar{x}_j (H_k e_i', H_k e_j') \leqslant \sum_i x_i \bar{x}_i = \| \Delta^k x \|^2$$

and $|\sum \lambda_i \omega_{e_i', e_i}(H_k)| = \sum \lambda_i$ (by slightly modifying H_k by operators of $\mathscr{L}(\mathscr{D})$, this shows (2)). Thus, for every $k \in \mathbb{N}$, $\varphi_k = \sum_{i=1}^{n_k} \lambda_i^k \omega_{e_i'^k, e_i^k}$ with

$$\| \Delta^k e_i'^k \| = \| e_i^k \| = 1 \qquad \text{and} \qquad \sum_{i=1}^{n_k} \lambda_i^k = \| \varphi_k \|_{\Delta^k} \,.$$

Thus

$$\varphi = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n_k} \omega_{(\lambda_i^k)^{1/2} e_i'^k, (\lambda_i^k)^{1/2} e_i^k} \right).$$

Now, given $A \in \mathscr{L}(\mathscr{D})$, the series

$$\sum_{k=1}^{\infty} \sum_{i=1}^{n_k} \| A (\lambda_i^k)^{1/2}\, e_i'^k \|^2$$

is σ-convergent, since using Lemma 1.1, for every n

$$\sum_{k=1}^{\infty} \sum_{i=1}^{n_k} \| \Delta^n (\lambda_i^k)^{1/2} e_i'^k \|^2$$

converges because of relation

$$\sum_{k=1}^{\infty} \| \varphi_k \|_{\Delta^n, \lambda} < +\infty$$

obtained from

$$\sum_{k=1}^{\infty} \| \varphi_k \|_{\Delta^k, \lambda} < +\infty.$$

Now

$$\sum_{k=1}^{\infty} \left(\sum_{i=1}^{n_k} \| (\lambda_i^k)^{1/2} e_i^k \|^2 \right) = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n_k} \lambda_i^k \right) = \sum_{k=1}^{\infty} \| \varphi_k \|_{\Delta^k} < +\infty$$

achieves the proof (see Definition 1.6(iv)).

2. Special $*$-Algebras and Examples

In this section, we examine particular $*$-algebras. Examples will be discussed and listed after a few abstract results. We introduce for convenience the following definition:

Definition 2.1. A ρ-closed $*$-algebra of countable algebraic dimension, with closed domain, is called hyperfinite.

Let us recall that algebras of countable algebraic dimension are countably dominated.

Proposition 2.1. *Let $\mathcal{A}$ be a $*$-algebra of countable algebraic dimension. Consider the following statements.*

(i) *$\mathcal{A}$ is hyperfinite.*

(ii) *For every $A \in \mathcal{A}^+$, $\mathfrak{N}_A$ is finite dimensional.*

(iii) *The ρ-topology of $\mathcal{A}$ is the finest locally convex topology on $\mathcal{A}$.*

(iv) *$\mathcal{A}$ is λ-closed.*

(v) *For every $A \in \mathcal{A}$, $\mathfrak{M}_A$ is finite dimensional.*

(vi) *The λ-topology of $\mathcal{A}$ is the finest locally convex topology on $\mathcal{A}$.*

Then, one has the implications

$$i \Leftrightarrow ii \Leftrightarrow iii \Rightarrow iv \Leftrightarrow v \Leftrightarrow vi.$$

Of course, if $\mathcal{A}$ is hyperfinite, then $\lambda = \rho$.

Proof. It is an easy consequence of the Baire theorem that a Banach space cannot be of countable algebraic dimension unless it is a finite-dimensional space. Therefore i $\Leftrightarrow$ ii (resp. iv $\Leftrightarrow$ v).

Clearly, ii $\Rightarrow$ iii (resp. v $\Rightarrow$ vi), since the ρ-topology (resp. the λ-topology) is the inductive limit of finite dimensional spaces. If iii (resp. iv) is true, then the bounded subset of $\mathfrak{A}$ are finite-dimensional: From Proposition 1.2 (resp. Proposition 1.6), this implies (i) (resp. iv).

COROLLARY 2.1. (existence of a predual). *Let $\mathfrak{A}$ be a hyperfinite $*$-algebra. Then, there exists a Frechet space $\mathfrak{A}_*$, whose strong dual is $\mathfrak{A}$. Moreover, if $\tilde{\mathfrak{A}}_*$ is a second Frechet space with strong dual $\mathfrak{A}$, then $\tilde{\mathfrak{A}}_*$ is topologically isomorphic to $\mathfrak{A}_*$.*

Proof. The first assertion follows from reflexivity of $\mathfrak{A}$; it suffices to take $\mathfrak{A}_* = \mathfrak{A}'$, endowed with strong topology (the strong topology agrees with $\sigma(\mathfrak{A}', \mathfrak{A})$ which is metrizable since $\mathfrak{A}$ is of countable algebraic dimension).

If $\tilde{\mathfrak{A}}_*$ is a second predual for $\mathfrak{A}$, then $\tilde{\mathfrak{A}}_*$ is reflexive ([10, p. 96]). Since the topology of $\tilde{\mathfrak{A}}''_*$ is the topology of uniform convergence on equicontinuous subsets of $\mathfrak{A}$, it necessarily agrees with the strong topology of $\mathfrak{A}'$. This achieves the proof.

It is clear from Proposition 2.1 that a hyperfinite algebra $\mathfrak{A}$ is a nuclear space, and that $(S, T) \to ST$ is continuous; hence, if f is a linear form on $\mathfrak{A}$, the map $T \in \mathfrak{A} \to f(T^*T)^{1/2}$ is continuous.

An application of Theorem 1 of [11] leads to

COROLLARY 2.2. *Let $\mathfrak{A}$ be a hyperfinite $*$-algebra, and f a state on $\mathfrak{A}$. Then, there is a standard measure space Z, a weakly measurable map $\xi \to f_\xi$ from Z into the set of extremal states of $\mathfrak{A}$, and a positive measure μ on Z (with $\mu(Z) = 1$) such that*

$$f = \int_Z f_\xi \, d\mu(\xi).$$

COROLLARY 2.3. *Let $\mathfrak{A}$ be a hyperfinite $*$-algebra; and $\mathfrak{A}^+$ the convex cone of positive elements of $\mathfrak{A}$. Then*

1° *for every $T \in \mathfrak{A}^+$, there exists $S \in \mathfrak{A}^-$ such that*

$$0 \leqslant S \leqslant T \qquad \textit{and} \qquad \dim \mathfrak{N}_S = 1,$$

2° *$\mathfrak{A}^+$ is the convex envelop of its extreme rays,*

3° *$\mathfrak{A}$ can be written*

$$\mathfrak{A} = \bigoplus_{n \in N} \mathfrak{N}_{A_n}$$

with $A_n \geqslant 0$ and $\dim \mathfrak{N}_{A_n} = 1$, for all $n \in \mathbb{N}$.

Let us note that unicity does not hold in 3°.

Proof. We recall that an element T of $\mathcal{A}^+$ belongs to some extremal ray of $\mathcal{A}^+$ if and only if the relation $0 \leqslant S \leqslant T$ implies $S = \lambda T$, for some $\lambda \geqslant 0$. Let $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathfrak{N}_{A_n}$, for certain $A_n \geqslant 0$. Then $\mathcal{A}^+ = \bigcup_{n\in\mathbb{N}} (\mathfrak{N}_{A_n} \cap \mathcal{A}^+) = \bigcup_{n\in\mathbb{N}} \mathfrak{N}^+_{A_n}$. Each set $\mathfrak{N}^+_{A_n}$ being a closed convex pointed cone of a finite dimensional space admits a compact (convex) base. The correspondence between extreme points (of the convex compact base) and extreme rays, the Krein–Milman theorem, the finite dimension of the base imply assertions 1° (note that an extreme ray of $\mathfrak{N}^+_{A_n}$, is an extreme ray of $\mathcal{A}^+$, and conversely). Thus, the set of extreme rays of $\mathcal{A}^+$ is the union, for $n \in \mathbb{N}$, of the set of extreme rays of $\mathfrak{N}^+_{A_n}$. Assertions 2° and 3° are therefore clear.

COROLLARY 2.4. *If $\mathcal{A}$ is a hyperfinite $*$-algebra, and $\mathcal{B}$ a $*$-subalgebra of $\mathcal{A}$, then $\mathcal{B}$ is hyperfinite.*

Proof. Obvious (note, however, that the domain $\mathcal{D}$ of $\mathcal{A}$ is not $\mathcal{B}$-closed: by definition, the domain of $\mathcal{B}$ is obtained by the processes of [19]).

We now recall a few facts about infinite tensor products. First, let $(\mathcal{D}_i)_{i\in\mathbb{N}}$ be a sequence of vector spaces, each $\mathcal{D}_i$ being dense in some Hilbert space $\mathfrak{h}_i$. We denote, for every $i\in\mathbb{N}$, by a_i an element of $\mathcal{D}_i$, with norm equal to 1. The subspace of the Hilbert space $\mathfrak{h} = \otimes_{(a_i)} \mathfrak{h}_i$ (incomplete tensor product), generated by elements $\otimes x_i$, with $x_i \in \mathcal{D}_i$ and $x_i = a_i$ for almost i, is dense in $\mathfrak{h}$, and will be noted simply $\otimes_i \mathcal{D}_i$. We consider now a sequence $(\mathcal{A}_i)_{i\in\mathbb{N}}$ of $*$-algebras, with closed domains $\mathcal{D}_i$ (dense in the Hilbert spaces $\mathfrak{h}_i$). For a family $(T_i)_{i\in\mathbb{N}}$ of operators, with $T_i \in \mathcal{A}_i$, and $T_i = \mathrm{Id}\,\mathfrak{h}_i$ for almost all i, one can define the operator $\otimes_i T_i$ to be the operator whose action on the element $\otimes_i x_i$ of $\otimes_i \mathcal{D}_i$ is $\otimes_i T_i x_i$.

The algebra $\mathcal{A}$ generated by such operators will be denoted $(\otimes_i \mathcal{A}_i, \otimes_i \mathcal{D}_i)$. Since we are mostly concerned with algebras with closed domains, by applying the method of [19] Lemma 2.6, we can extend the domain $\otimes_i \mathcal{D}_i$ to a new domain $\mathcal{D}$ (denoted $\mathcal{D} = \otimes_{a_i} \mathcal{D}_i$) such that $\mathcal{D}$ becomes $\mathcal{A}$-closed. By the infinite tensor product of $*$-algebras $\mathcal{A}_i$, we mean from now on the couple $(\mathcal{A}, \mathcal{D})$, noted simply $\otimes_{i=1}^{\infty} \mathcal{A}_i$.

PROPOSITION 2.2. 1° *If $(\mathcal{A}_i)_{1\leqslant i\leqslant n}$ is a finite set of hyperfinite $*$-algebras, then $\otimes_{i=1}^{n} \mathcal{A}_i$ is a hyperfinite $*$-algebra.*

2° *If $(\mathcal{A}_i)_{i\in\mathbb{N}}$ is a sequence of hyperfinite $*$-algebras, such that $\mathfrak{N}(1; \mathcal{A}_n) = \mathbb{C}\,\mathrm{Id}$ for almost all n, then the infinite tensor product $\otimes_{n=0}^{\infty} \mathcal{A}_n$ is a hyperfinite $*$-algebra.*

In the above formulation, $\mathfrak{N}(1; \mathcal{A}_n)$ denotes the set of bounded operators of $\mathcal{A}_n$.

Proof. For assertion 1°, it suffices to treat the case of two hyperfinite $*$-

algebras $\mathcal{A}$ and $\mathcal{B}$. Let A_i (resp. B_i) be operators of $\mathcal{A}^+$ (resp. $\mathcal{B}^+$) such that $\mathcal{A} = \bigcup_{i=1}^{\infty} \mathfrak{N}_{A_i}$ (resp. $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathfrak{N}_{B_i}$). We prove that

$$\mathcal{A} \otimes \mathcal{B} = \bigcup_{i=1}^{\infty} \mathfrak{N}_{A_i \otimes 1 + 1 \otimes B_i} .$$

For this, it suffices to consider elements of the form $S \otimes T$ ($S \in \mathcal{A}$, $T \in \mathcal{B}$). From the relation

$$S \otimes T = (S \otimes 1)(1 \otimes T), \qquad S, T \in \mathcal{A},$$
$$(AB\varphi, \varphi)| \leqslant \tfrac{1}{2}((A^*A\varphi, \varphi) + (B^*B\varphi, \varphi)), \qquad A, B \in \mathcal{A},$$

we have to treat elements of the form $S \otimes 1$. But, for some i,

$$S \otimes 1 \in \mathfrak{N}_{A_i \otimes 1} \subset \mathfrak{N}_{A_i \otimes 1 + 1 \otimes B_i}$$

which etablishes our assertion.

Now let $\sum_{\lambda \in \Lambda} S_\lambda \otimes T_\lambda$ (Λ finite set) be an element of $\mathfrak{N}_{A_i \otimes 1 + 1 \otimes B_i}$. Then for every vector of the form $\varphi \otimes \psi$, we have

$$\left| \sum_{\lambda} (S_\lambda \varphi, \varphi)(T_\lambda \psi, \psi) \right| \leqslant M((A_i\varphi, \varphi)(\psi, \psi) + (\varphi, \varphi)(B_i\psi, \psi)$$

for some $M > 0$.

Thus, for every vector φ, the operator $\sum_\lambda (S_\lambda \varphi, \varphi) T_\lambda$ belongs to $\mathfrak{N}_{B_i}$ (one can assume $A_i \geqslant 1$ and $B_i \geqslant 1$ for all $i \in \mathbb{N}$). By polarization, operators of the form $\sum_{i,j} \sum_\lambda (S_\lambda \varphi_i, \varphi_j) T_\lambda$ (for a finite set of vectors φ_i, φ_j) belong to $\mathfrak{N}_{B_i}$. This last space being finite-dimensional, $\sum_\lambda f(S_\lambda)\, T_\lambda \in \mathfrak{N}_{B_i}$ for every (continuous) linear form $f \in \mathcal{A}'$. Choosing the S_λ linearly independent, and using the Hahn–Banach theorem, we conclude $T_\lambda \in \mathfrak{N}_{B_i}$, for every $\lambda \in \Lambda$. In the same way, by symmetry, S_λ belongs to $\mathfrak{N}_{A_i}$. Thus

$$\mathfrak{N}_{A_i \otimes 1 + 1 \otimes A_i} \subset \mathfrak{N}_{A_I \otimes 1} + \mathfrak{N}_{1 \otimes B_i} .$$

This achieves part 1°.

For assertion 2°, it suffices to imitate the preceding proof.

Weyl Algebras

We denote by $\mathcal{S}$ the space of C^∞ rapidly decreasing functions of the real line. $\mathcal{S}$ is dense in the Hilbert space $\mathfrak{h} = L^2(\mathbb{R}; dx)$ (dx Lebesgue measure), and invariant under the operators of the complex algebra $\mathcal{A}$, generated by p, q, with

$$pf = \frac{d}{dx} f, \qquad qf = xf, \quad f \in \mathcal{S}.$$

Moreover, $\mathcal{S}$ is $\mathcal{Cl}$-closed and, by [16], $\mathcal{Cl} = \bigcup_{n\geqslant 0} \mathfrak{N}_{(1-\Delta)^n}$ with $\Delta = p^2 - q^2$. We refer to [6] for a detailled algebraic study of $\mathcal{Cl}$.

PROPOSITION 2.3. *$\mathcal{Cl}$ is hyperfinite.*

Proof. Let V be the linear algebraic space, spanned by Hermite functions

$$\varphi_k(x) = (\pi^{1/2}2^k k!)^{-1/2} (-1)^k e^{x^2/2} \frac{d^k}{dx^k} (e^{-x^2}), \qquad x \in \mathbb{R}.$$

Since

$$\begin{aligned} p\varphi_k &= 2^{-1/2}(k^{1/2}\varphi_{k-1} - (k+1)^{1/2}\varphi_{k+1}), \\ q\varphi_k &= 2^{-1/2}(k^{1/2}\varphi_{k-1} + (k+1)^{1/2}\varphi_{k+1}), \end{aligned}$$

V is stable under operators of $\mathcal{Cl}$. Moreover, V being dense in $\mathcal{S}$ endowed with its Fréchet topology (defined by seminorms $\varphi \to \|(1-\Delta)^n\varphi\|$, $n \in \mathbb{N}$) the operators of $\mathcal{Cl}$ are completely determined by their restrictions to V (being continuous for this Fréchet topology). We introduce

$$P = 2^{-1/2}(p + q) \qquad Q = 2^{-1/2}(q - p)$$

and note that $\mathcal{Cl}$ is graded by $\mathcal{Cl}^r = \sum_{j-i=r} \mathbb{C}\, P^i Q^j$, $r \in \mathbb{Z}$ [6]. Let $u = \sum_r u_r$ be an element of $\mathfrak{N}_{(1-\Delta)^n}$. Then

$$(u\varphi_k\,, \varphi_{k+r}) = (u_r\varphi_k\,, \varphi_{k+r})$$

with $\varphi_k = 0$ iff $k > 0$. Thus, u_0 belongs to $\mathfrak{N}_{(1-\Delta)^n}$ since

$$\begin{aligned} \left|\left(u_0 \sum_k \lambda_k\varphi_k\,, \sum_k \lambda_k\varphi_k\right)\right| &= \left|\sum_k |\lambda_k|^2 (u\varphi_k\,, \varphi_k)\right| \\ &\leqslant M_1 \left((1-\Delta)^n \sum_k \lambda_k\varphi_k\,, \sum_k \lambda_k\varphi_k\right) \end{aligned}$$

where $M_1 < +\infty$. If $r \neq 0$, the consideration of

$$((u - u_0)\,\varphi_k + \varphi_{k+r}\,, \varphi_k + \varphi_{k+r})$$

and

$$((u - u_0)\,\varphi_k + i\varphi_{k+p}\,, \varphi_k + i\varphi_{k+r})$$

leads to the estimate

$$\begin{aligned} |(u_r\varphi_k\,, \varphi_{k+r})| &\leqslant M_2[((1-\Delta)^n\,\varphi_k\,, \varphi_k) + ((1-\Delta)^n\,\varphi_{k+r}\,, \varphi_{k+r})] \\ &= M_3(k)\,(k+1)^n, \end{aligned}$$

where $M_2 < +\infty$, and $M_3(k)$ positive bounded function of k. Thus, putting

$$u_r = \sum_{i=0}^{l} \alpha_i P^i Q^{i+r} \qquad \alpha_i \in \mathbb{C}$$

we conclude that

$$\sum_{i=0}^{l} \alpha_i \left[\frac{(k+i+r)!}{k!} \frac{(k+i+r)!}{(k+r)!}\right]^{1/2} \leqslant M_3(k)(1+k)^n.$$

Then, necessarily $l \leqslant (2n-r)/2$, showing that $\mathfrak{N}_{(1-\Delta)^n}$ is finite-dimensional.

COROLLARY 2.5. *Let $\mathfrak{A}_m$ be the complex associative algebra generated by operators p_i and q_i of $L^2(\mathbb{R}^m, dx)$, defined by*

$$p_i f = (\partial/\partial x_i) f, \qquad q_i(f) = x_i f,\ 1 \leqslant i \leqslant m,$$

f belonging to $\mathcal{S}(\mathbb{R}^m)$, the space of C^∞ rapidly decreasing functions in m variables. Then, $\mathfrak{A}_m$ is hyperfinite.

Proof. The proof is an immediate consequence of Proposition 2.2, 1°. A similar corollary holds for an infinite tensor product.

We can now turn to a more complete description of Corollary 2.3. For simplicity we denote I the set

$$I = \{u \in \mathfrak{A};\ u = \lambda p + \mu q + \nu \text{ with } u \text{ symmetric, } \lambda, \mu, \nu \in \mathbb{C}\}.$$

PROPOSITION 2.4. *$\mathfrak{A}$ can be written $\mathfrak{A} = \sum_{k\geqslant 0\ u\in I} \mathfrak{N}_{u^{2k}}$. Moreover, for every $u \in I$, and $k \geqslant 0$,*

$$\dim \mathfrak{N}_{u^{2k}} = 1.$$

Proof. Let u be an element of I: u being symmetric, there exists a unitarily implementable automorphism Φ of $\mathfrak{A}$ such that :

$$\Phi(ip) = u \qquad [2].$$

Then we just have to prove that $\dim \mathfrak{N}_{(ip)^{2k}} = 1\ \forall k \geqslant 0$. Let $v = \sum_{i,j} \lambda_{ij} q^j p^i$, $\lambda_{ij} \neq 0$, be an element of $\mathfrak{N}_{(ip)^{2k}}$. For all unitarily implementable automorphisms ψ of $\mathfrak{A}$ such that $\psi(p) = p$, we have:

$$\psi(v) \in \mathfrak{N}_{(ip)^{2k}} \qquad \text{and} \qquad \psi(v) = \sum_{i,j} \lambda_{ij} (\psi(q))^j\, p^i.$$

Now, the space generated by the $\psi(q)$, ψ a such automorphism, is infinite-dimensional since the degree of $\psi(q)$ can be arbitrarily large. So if there exists an index $j > 0$ in $v = \sum \lambda_{ij} q^j p^i$, $\mathfrak{N}_{(ip)^{2k}}$ is infinite-dimensional. By Proposition 2.1 this is impossible. Thus we have:

$$v = \sum_i \lambda_i p^i \in \mathfrak{N}_{(ip)^{2k}} .$$

But after a Fourier transform, this relation gives

$$\sum_i \lambda_i \int_{\mathbb{R}} t^i \, |f(t)|^2 \, dt \leqslant M \int_{\mathbb{R}} t^{2k} \, |f(t)|^2 \, dt \qquad \forall f \in \mathscr{S}, \quad M < +\infty.$$

Therefore, choosing the function f with support in $[n, n+1]$, next in $[1/n+1, 1/n]$, we conclude that $i = 2k$ and $\mathfrak{N}_{(ip)^{2k}}$ is one-dimensional. Finally, let $\mathscr{B}$ be the space $\sum_{k \geqslant 0} {}_{u \in I} \mathfrak{N}_{u^{2k}}$. For any integer n, $(ip)^n$ belongs to $\mathscr{B}$ since $(ip)^{2k}$ and $(ip+1)^{2k}$ belong to $\mathscr{B}$ for all k. For the same reason, q^n and $v_\lambda = (ip + \lambda q)^n$ $(\lambda \in \mathbb{R})$ also belong to $\mathscr{B}$. But the v_λ for $n+1$ different values of λ, generates a vector space which contains all the monomials $q^i p^{n-i}$, the determinant of the system:

$$F_\lambda(q^i p^{n-i}) = v_\lambda$$

being a van der Monde determinant, which proves the proposition.

Unitary Representations of Lie Groups

Let G be a real Lie group with Lie algebra $\mathfrak{g}$, and U a continuous unitary representation of G in a separable Hilbert space $\mathfrak{h}$. We denote by $\mathscr{D}$ the space of differentiable vectors of U, by dU the differential of U; dU is a representation of $\mathfrak{g}$, and operators $dU(X)$ $(X \in \mathfrak{g})$ are clearly defined on $\mathscr{D}$. We also introduce $\mathfrak{U}$ the complex enveloping algebra of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$, and $\mathscr{O}\!\!\!l$ the complex $*$-algebra $dU(\mathfrak{U})$ ($\mathscr{O}\!\!\!l$ acts on the $\mathscr{O}\!\!\!l$-closed domain $\mathscr{D}$).

PROPOSITION 2.5. *If G is nilpotent and U irreducible, $\mathscr{O}\!\!\!l$ is hyperfinite.*

Proof. Indeed, $\mathscr{O}\!\!\!l$ is the algebra studied in Corollary 2.5.

PROPOSITION 2.6. *Let U be an irreducible unitary representation of the group $G = SL(2, \mathbb{R})$. Then, the algebra $\mathscr{O}\!\!\!l$ is hyperfinite.*

Proof. Indeed, $\mathscr{O}\!\!\!l$ is an algebra $\mathscr{B}_\lambda$ studied in [7]; with notations of this paper, we get a basis of $\mathscr{O}\!\!\!l$ of the form

$$\{dU(Y^n F^p), dU(Y^m F^q), n, p, q \geqslant 0, m > 0\}.$$

Moreover, we can find in $\mathfrak{h}$ an orthonormal basis φ_k satisfying

$$\begin{aligned} dU(Y)\varphi_k &= \lambda_k\varphi_k\,, \\ dU(F)\varphi_k &= \alpha_k\varphi_{k+1}\,, \\ dU(G)\varphi_k &= \beta_k\varphi_{k-1}\,, \\ (1-\Delta)\varphi_k &= (q+2\lambda_k{}^2)\varphi_k\,, \end{aligned}$$

with $k \in \mathbb{Z}$ or $\mathbb{N}$, according to the case, q a constant and λ_k, α_k, β_k numbers of order k when $|k|$ goes to infinity. Now it suffices to adjust the proof of Proposition 2.3.

Of course, for irreducible representations of compact groups, $\mathcal{A}$ is hyperfinite. Some insight in factorial representations can be obtained by

LEMMA 2.1. *Let U be a factorial representation of a group G of type I. Then, there exists a unique unitary irreducible representation U_0 of G such that $\mathcal{A} = dU(\mathfrak{U})$ is isomorphic to $dU_0(\mathfrak{U})$.*

Proof. Let $\mathfrak{h} = \int_X^{\oplus} \mathfrak{h}(\xi)\, d\mu(\xi)$, $U = \int_X^{\oplus} U(\xi)\, d\mu(\xi)$ a decomposition of U into a direct integral of irreducible representations of G. By [5], the representation $U(\xi)$ is almost everywhere equivalent to an irreducible unitary representation U_0 of G. Thus, we can write [4, p. 165, 193]

$$\mathfrak{h} = \mathfrak{h}_0 \otimes L^2(X, d\mu), \qquad U(g) = U_0(g) \otimes 1, \qquad g \in G,$$

$\mathscr{D}_0$ being the space of differentiable vectors for U_0, we get by differentiation

$$dU = dU_0 \otimes 1 \text{ on } \mathscr{D}_0 \otimes L^2(X, d\mu) \qquad \text{(algebraic tensor product).}$$

Moreover, $\mathscr{D}_0 \otimes L^2(X, d\mu)$ is dense in $\mathscr{D}$ for the topology of $\mathscr{D}$ since this space is invariant under $U(G)$ [18]. Therefore, the map $T \to T \otimes 1$ of $dU_0(\mathfrak{U})$ onto $dU(\mathfrak{U})$ is an isomorphism, bicontinuous by [13]. Q.E.D.

It is worth noting that for nonfactorial representations, the structure of $\mathcal{A} = dU(\mathfrak{U})$ can be complicated: for instance, certain unitary representations of abelian groups leads to algebras of functions, showing the great variety of situations. Nevertheless,

PROPOSITION 2.7. *Let G be the Heisenberg group (defined in [22]), and $Z \neq 0$ a central element of $\mathfrak{g}$. Assume that one of the following hypotheses is true*

(i) *$dU(Z)$ is an unbounded operator,*

(ii) *the spectrum of $dU(Z)$ is finite, and different from $\{0\}$.*

Then, $\mathcal{A}$ is hyperfinite.

Proof. Let $\{X_1, \ldots, X_m, Y_1, \ldots, Y_m, Z\}$ be a basis of $\mathfrak{g}$ satisfying the relations:

$$[X_i, Y_j] = \delta_{ij} Z \qquad [X_i, X_j] = [Y_i, Y_j] = [X_i, Z] = [Y_j, Z] = 0.$$

We decompose U in $U_1 \oplus U_2$ where U_1 is a direct integral of infinite dimensional irreducible representations of G and U_2 is a direct integral of characters of G. If (i) is true, we can write:

$$U_1 = \int_X^{\oplus} U_\lambda \, d\mu(\lambda),$$

where U_λ is factorial, X is the spectrum of $-i\, dU_1(Z)$ and μ is a positive Radon measure on $\mathbb{R}$. Moreover, U_λ can be expressed as $U_\lambda = \tilde{U}_\lambda \otimes 1$ with $\tilde{U}_\lambda$ irreducible infinite-dimensional of G (Lemma 2.1): $\tilde{U}_\lambda$ acts in $L^2(\mathbb{R}^m)$ and we have:

$$d\tilde{U}_\lambda(X_i) = \partial/\partial x_i, \qquad d\tilde{U}_\lambda(Y_i) = i\lambda x_i, \qquad d\tilde{U}_\lambda(Z) = i\lambda.$$

Let V be the space spanned by the Hermite functions, and W the space $(\otimes^m V) \otimes \mathscr{K}$ where $\mathscr{K}$ is the subspace of functions of compact support in $L^2(\mathbb{R}, d\mu)$: W is stable under $dU_1(\mathfrak{g})$. Since $dU_1|_W$ is faithful ($\operatorname{Ker} dU_1 \cap \mathbb{C}[Z] = \{0\}$ and [7, p. 148]) we consider only $dU_1|_W(\mathfrak{U}) = \mathfrak{A}_1$.

First, we prove that if $A = \int^\oplus A_\lambda \, d\mu$ and $B = \int^\oplus B_\lambda \, d\mu$ are two elements of $\mathfrak{A}_1$ such that: $|(A\psi, \psi)| \leqslant (B\psi, \psi)$ for all ψ of W, then μ almost everywhere:

$$|(A_\lambda \varphi, \varphi)| \leqslant (B_\lambda \varphi, \varphi) \tag{1}$$

for all φ of $\otimes^m V$.

Indeed if K is a compact set in $\mathbb{R}$, f an element of $L^2(K, d\mu)$ and (φ_k) a basis of $\otimes^m V$, we have for all integers k and k':

$$\left| \int_K (A_\lambda f(\lambda)\, \varphi_k, f(\lambda)\, \varphi_{k'})\, d\mu(\lambda) \right| \leqslant \int_K (B_\lambda f(\lambda)\, \varphi_k, f(\lambda)\, \varphi_{k'})\, d\mu(\lambda),$$

Thus, almost everywhere:

$$|(A_\lambda \varphi_k, \varphi_{k'})| \leqslant (B_\lambda \varphi_k, \varphi_{k'}) \qquad \forall k, k',$$

showing (1). Now let $dU_1(u)$ be an element of $\mathfrak{N}_{(1-\Delta)^n}$. We can write

$$u = \sum_{I,J,k} \alpha_{I,J,k} (X)^I (Y)^J Z^k,$$

I, J being elements of $\mathbb{N}^m$ ($\mathbb{N}$ the integers), k an integer, and $(X)^I = X_1^{i_1} \cdots X_m^{i_m}$, $|I| = i_1 + i_2 + \cdots + i_m$. Thus

$$dU_1(u) = \sum_{I,J,k} i^k \alpha_{IJk} \left(\frac{\partial}{\partial x_i} \right)^I (ix_j)^J \otimes \lambda^{|J|+k_1}.$$

By the same method:

$$(1 - \Delta)^n = \left[1 \otimes 1 - \sum_i \left(\frac{\partial}{\partial x_i}\right)^2 \otimes 1 + \sum_j (x_j)^2 \otimes \lambda^2 1 + 1 \otimes \lambda^2\right]^n.$$

Then almost everywhere:

$$\left| \sum_{I,J,k} \alpha_{I,J,k} \left(\left(\frac{\partial}{\partial x_i}\right)^I (ix_j)^J \lambda^{|J|+k}\varphi, \varphi\right)\right|$$
$$\leqslant M \left(\left(1 + \lambda^2 - \sum_i \left(\frac{\partial}{\partial x_i}\right)^2 + \lambda^2 \sum_j x_j^2\right)^n \varphi, \varphi\right), \qquad \varphi \in \bigotimes^m V.$$

And so $|I| + |J| \leqslant 2nm$ following the extimations of Proposition 2, and this corollary. Now if φ belongs to $\otimes^m V$, $\|\varphi\| = 1$ for all functions f of $\mathscr{K}$, we have almost everywhere:

$$\left| \sum_{IJk} i^k \alpha_{IJk} \left(\frac{\partial\varphi}{\partial x_i}, \varphi\right)^I (ix_j\varphi, \varphi)^J (\lambda^{|J|+k} f(\lambda), f(\lambda)) \right| \leqslant M(n, \varphi)(P(\lambda) f(\lambda), f(\lambda)),$$

where $M(n, \varphi)$ is finite and $P(\lambda)$ is a polynomial of degree $2n$. If λ goes to infinity, we obtain: $|J| + k \leqslant 2n$, and $\mathfrak{N}_{(1-\Delta)^n}$ is finite-dimensional.

Now assume (ii) is true, and denote $\{\lambda_1, \ldots, \lambda_r\}$ the spectrum of $dU(Z)$. It is easy to see that $\mathfrak{A}$ is linearly spanned by the monomial $X^I Y^J Z^k$ with $0 \leqslant k \leqslant r$, I and J being elements of $\mathbb{N}^m$. We can decompose U in:

$$U = U_{\lambda_1} \oplus \cdots \oplus U_{\lambda_r}$$

where U_{λ_i} is the factorial representation of G considered in the first part of the proof. If $u = \sum \alpha_{I,J,k} X^I Y^J Z^k$ belongs to $\mathfrak{N}_{(1-\Delta)^n}$, then

$$|(dU_\lambda(u)\psi, \psi)| \leqslant M((1 - \Delta)^n \psi, \psi) \qquad \text{for} \quad \psi \in \bigotimes^m V,$$

hence $|I| + |J| \leqslant 2nm$. Q.E.D.

Topology of $\mathscr{D}$ and Structure of $\mathfrak{A}$

The aim of this Section is to show that there is no link between nuclearity of $\mathscr{D}$ and hyperfiniteness of $\mathfrak{A}$, even for enveloping algebras (viewed in some representation). In fact, this is not too surprising since if $\mathfrak{A}$ is written $\mathfrak{A} = \bigcup_{i \in I} \mathfrak{M}_{A_i}$, then the topology of $\mathscr{D}$ admits seminorms $x \to \| A_i x \|$, and the set of A_i, $i \in I$, does not necessarily reflect the algebraic structure of $\mathfrak{A}$.

LEMMA 2.2. *Let E_2 be the two-dimensional Euclidean group, and U the representation of E_2 defined in $L^2(T, dx)$ (T torus, dx Lebesgue measure) by*

$$(U(\alpha, \beta)f)(t) = e^{ir\mathrm{Re}(\beta e^{-it})} f(t - \alpha)$$

α being real ($0 \leqslant \alpha \leqslant 2\pi$), β complex being a parametrization of E_2 [22]. Then

1° *$\mathcal{A} = dU(\mathfrak{U})$ is not hyperfinite,*

2° *the space $\mathcal{D}$ of differentiable vectors of U is nuclear.*

Proof. $\mathcal{D}$ is the space of C^∞ functions on the torus [22] since $\mathfrak{g}$ is three-dimensional with basis (X, Y_1, Y_2) satisfying

$$(dU(X)f)(t) = -\frac{d}{dt}f,$$
$$(dU(Y_1)f)(t) = ir\cos t\, f(t),$$
$$(dU(Y_2)f)(t) = ir\sin t\, f(t).$$

Thus, $(1 - \Delta) = 1 - d^2/dt^2 + r^2$.

The space $\mathcal{D}$, endowed with seminorms $f \to \|(d^2/dt^2 + 1)^n f\|$ is nuclear $(1 - \Delta)^{-1}$ being a nuclear operator on $L^2(T)$. However, the algebra $\mathcal{A}$ is not hyperfinite, since the algebra $\mathbb{C}[Y_1]$ of polynomials in Y_1 is included in $\mathfrak{N}_{(1-\Delta)}$.

LEMMA 2.3. *Let G be the affine group of the real line, and U the representation of G defined in $L^2(\mathbb{R}, dx)$ by*

$$(U(\alpha, \beta)f)(t) = e^{i\beta t^2}\alpha^{1/4}f(\alpha^{1/2}t), \qquad t \in \mathbb{R},$$

$\alpha > 0$ and β real being a parametrization of G.

Then

1° *$\mathcal{A} = dU(\mathfrak{U})$ is hyperfinite,*

2° *the space $\mathcal{D}$ of differentiable vectors of U is not nuclear.*

Proof. G admits two irreducible infinite-dimensional unitary representations U^+ and U^- (we refer to [22] for details and notations), U^+ being defined on $L^2(\mathbb{R}, dx)$ by

$$(U^+(\alpha, \beta)\varphi)(x) = e^{i\beta e^x}\varphi(x + \operatorname{Log}\alpha), \qquad x \in \mathbb{R},$$

and U is unitarily equivalent to $U^+ \oplus U^+$: indeed, U splits in $U|_{L^2[0,\infty[} \oplus U|_{L^2]-\infty,0]}$ and these two components are unitarily equivalent, since the unitary operator V: $L^2(-\infty, 0) \to L^2(0, +\infty)$, $(V(f))(t) = f(-t)$ is an interwining operator. Also, $U|_{L^2[0,\infty)}$ is equivalent to U^+ because W: $L^2(0, +\infty) \to L^2(\mathbb{R})$ $((Wf)(x) = f(e^{2x})\, e^x/2^{1/2}))$ is an interwining operator (for $U|_{L^2(0,\infty)}$ and U^+). Now, $\mathfrak{g}$ is two-dimensional, with basis (X, Y) satisfying

$$[X, Y] = Y,$$
$$dU(X) = \tfrac{1}{2}t(d/dt) + \tfrac{1}{4},$$
$$dU(Y) = it^2.$$

Since f belongs to $\mathscr{D}$ if and only if the restrictions of f to $(-\infty, 0)$ and $(0, +\infty)$ are in $\mathscr{D}$, $\mathscr{D}$ is isomorphic to the topological direct sum $\mathscr{D}^- \oplus \mathscr{D}^+$, where $\mathscr{D}^+$ is the space of differentiable vectors for U^+. This sum is not nuclear because the Laplacian Δ of G is such that $dU^+(1 - \Delta)^{-1}$ is not compact, G being not liminar [17, 22].

Finally, the space $\mathscr{S}$ of C^∞ rapidly decreasing functions of the real line is dense in $\mathscr{D}$ [1] and so $\mathfrak{U}$ is isomorphic to a subalgebra of the Weyl algebra considered in the first part. Thus $\mathfrak{U}$ is hyperfinite.

A Situation with $\lambda \neq \rho$

Let $\mathfrak{h}$ be a separable infinite-dimensional Hilbert space, $(e_n\ ;\ n \in \mathbb{N})$ a complete orthonormal system of $\mathfrak{h}$. We note $\mathscr{D}$ the space algebraically spanned by the e_n and $\mathscr{A}$ the $*$-algebra generated by operators T and U (defined on $\mathscr{D}$) with

$$Te_n = ne_n, \qquad n \in \mathbb{N},$$
$$Ue_n = e_{n+1}, \qquad n \in \mathbb{N}.$$

$\mathscr{A}$ is countably dominated: more precisely, $\mathscr{A} = \bigcup_{p\geqslant 0} \mathfrak{N}_{T^p}$. Indeed an element of $\mathscr{A}$ is a sum of polynomials of the form

$$u(I, J, K) = \lambda_{I,J,K} U^{i_1}T^{j_1}(U^*)^{k_1}\, U^{i_2}T^{j_2}(U^*)^{k_2} \cdots U^{i_q}T^{j_q}(U^*)^{k_q},$$

where $I = \{i_1, ..., i_q\}$, $J = \{j_1, ..., j_q\}$, $K = \{k_1, ..., k_q\}$ are elements of $\mathbb{N}^q$ ($\mathbb{N}$ the positive integers). By induction, we prove that

$$u(I, J, K)e_n = P_{IJK}(n)e_{n+m(I,J,K)} \qquad \forall n,$$

where P_{IJK} is a polynomial of n, $m(I, J, K)$ an integer depending only on IJK. If p is the degree of P_{IJK}, we have an estimation

$$\| u(I, J, K)e_n \| = | P_{IJK}(n)| \leqslant M \| T^p e_n \| \qquad \forall n$$

where M is a finite constant. Thus for all x of $\mathscr{D}$:

$$\| u(I, J, K)x \| \leqslant M \| T^p x \|.$$

Now, the set $\mathscr{B} = \{(U^*)^k T,\ k \text{ integer},\ k \geqslant 0\}$ is bounded for the λ-topology. Indeed,

$$U^{*k}Te_n = 0 \text{ if } n \leqslant k \qquad \text{and} \qquad U^{*k}Te_n = ne_{n-k} \text{ if } k < n,$$

which proves

$$\| U^{*k}\, Tx \| \leqslant \| Tx \| \qquad \forall x \in \mathscr{D}.$$

If $\lambda = \rho$, the $*$-operation is continuous for the λ-topology and $\mathscr{B}^*$ is bounded. But for any $M > 0$, any positive integer n and p, we can find k such that

$$\| TU^k e_n \| = \|(k + n)e_{n+k}\| = k + n > Mn^p = M \| T^p e_n \|,$$

which proves that the element TU^k of $\mathscr{B}^*$ is not in $\mathfrak{M}_{T^p}$. Following Proposition 16 we conclude that $\mathscr{B}^*$ is not bounded. Q.E.D.

ACKNOWLEDGMENTS

The authors would like to thank Professor M. Flato for helpful advices and suggestions of many improvements; and Daniel Sternheimer for his kind interest in this work.

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